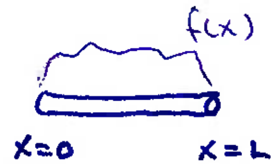
Heat Eq. (part 2)



last time: $u_t = k u_{xx}$ $\quad 0 < x < L \quad t > 0$

$u(0,t) = 0$ left end temp = 0

$u(L,t) = 0$ right end temp = 0

$u(x,0) = f(x)$ initial heat profile

$$u(x,t) = X(x)\,T(t)$$

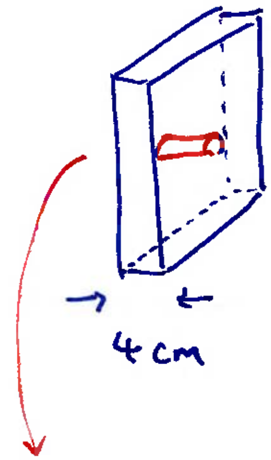
$X(x) = \sin\left(\frac{n\pi x}{L}\right)$ spatial solution

$T(t) = e^{-kn^2\pi^2 t/L^2}$ temporal solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

example : copper slab thickness 4 cm. $K = 1.15 \ cm^2/s$



entire interior is heated uniformly ~~too~~ to 100°C at $t=0$

then both faces (Left/right) are kept at 0°C for all t

"core sample"

entire plate is same material and uniformly heated

→ heat only flows left or right (1-D Heat Eq.)

solution (from last page)

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-1.15 n^2 \pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right)$$

at $t=0$, $f(x) = 100$

$$C_n = \frac{2}{4}\int_0^4 100 \sin\left(\frac{n\pi x}{4}\right) dx = \frac{200\left[1-(-1)^n\right]}{n\pi}$$

the time solution : $t \to \infty$, $u \to 0$ (heat flows out since both ends are cold)

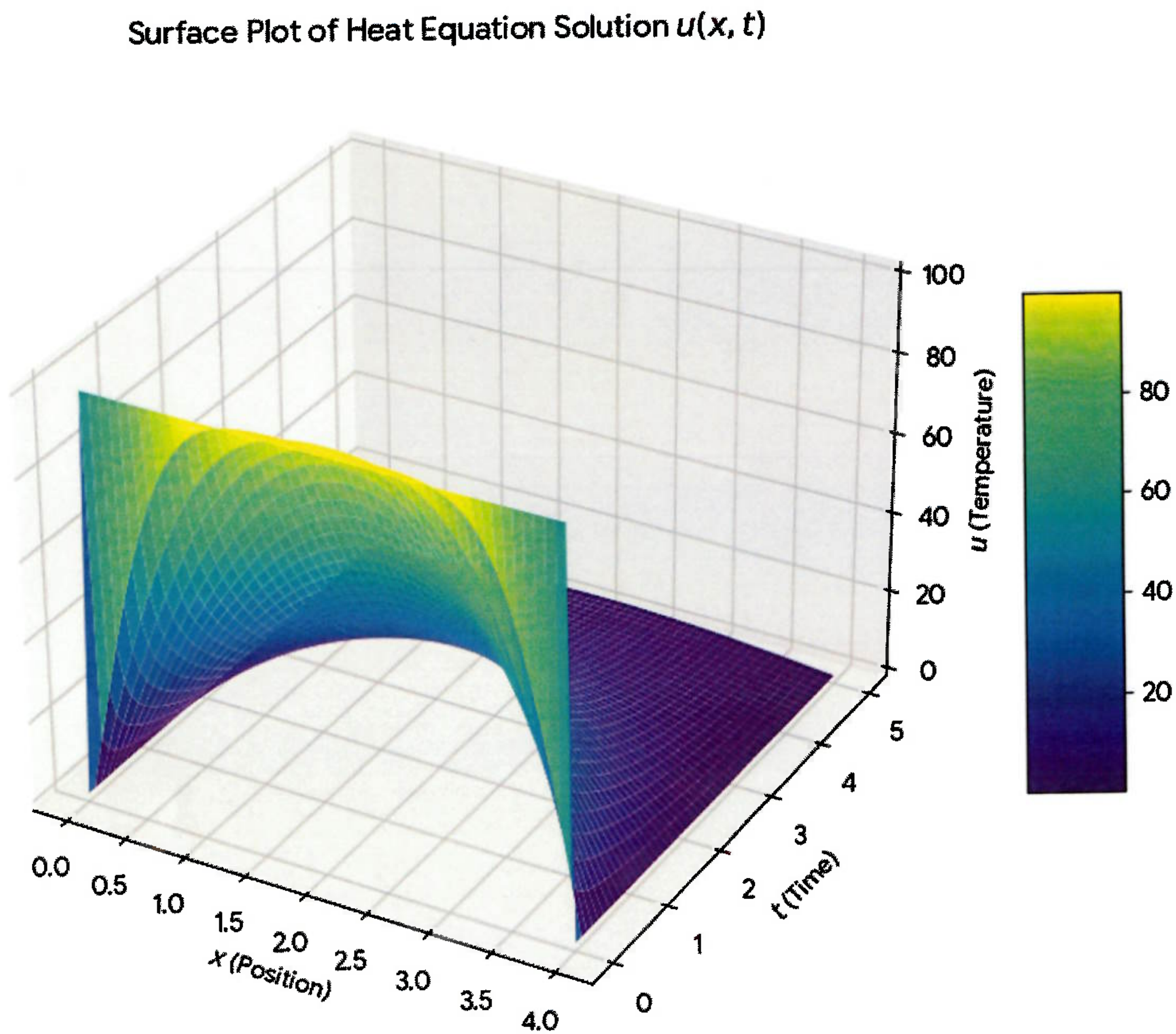
space solution: periodic because left end and right end are 0 °C

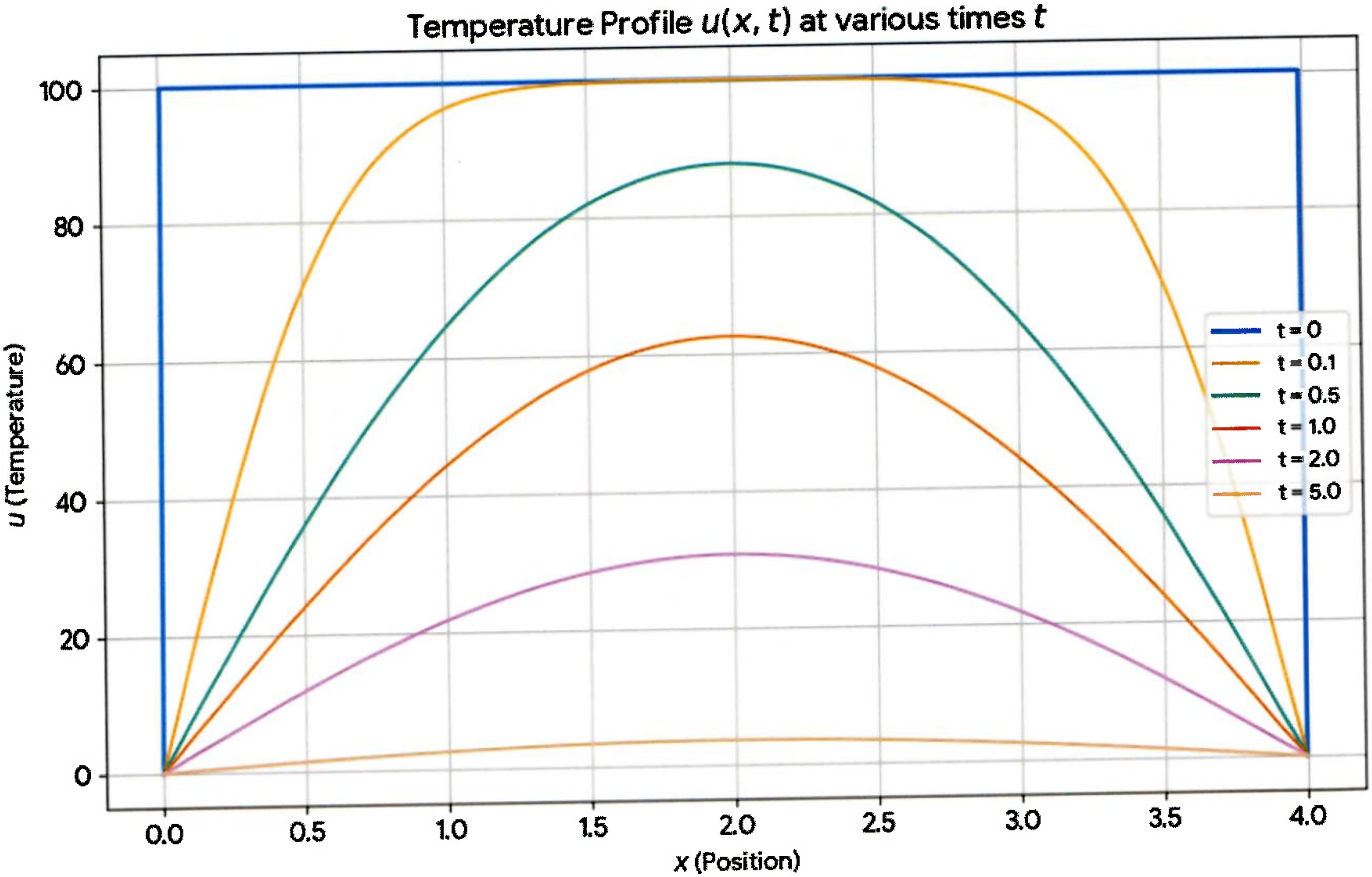
what is the temp. at mid point $(x=2)$ 3 seconds later?

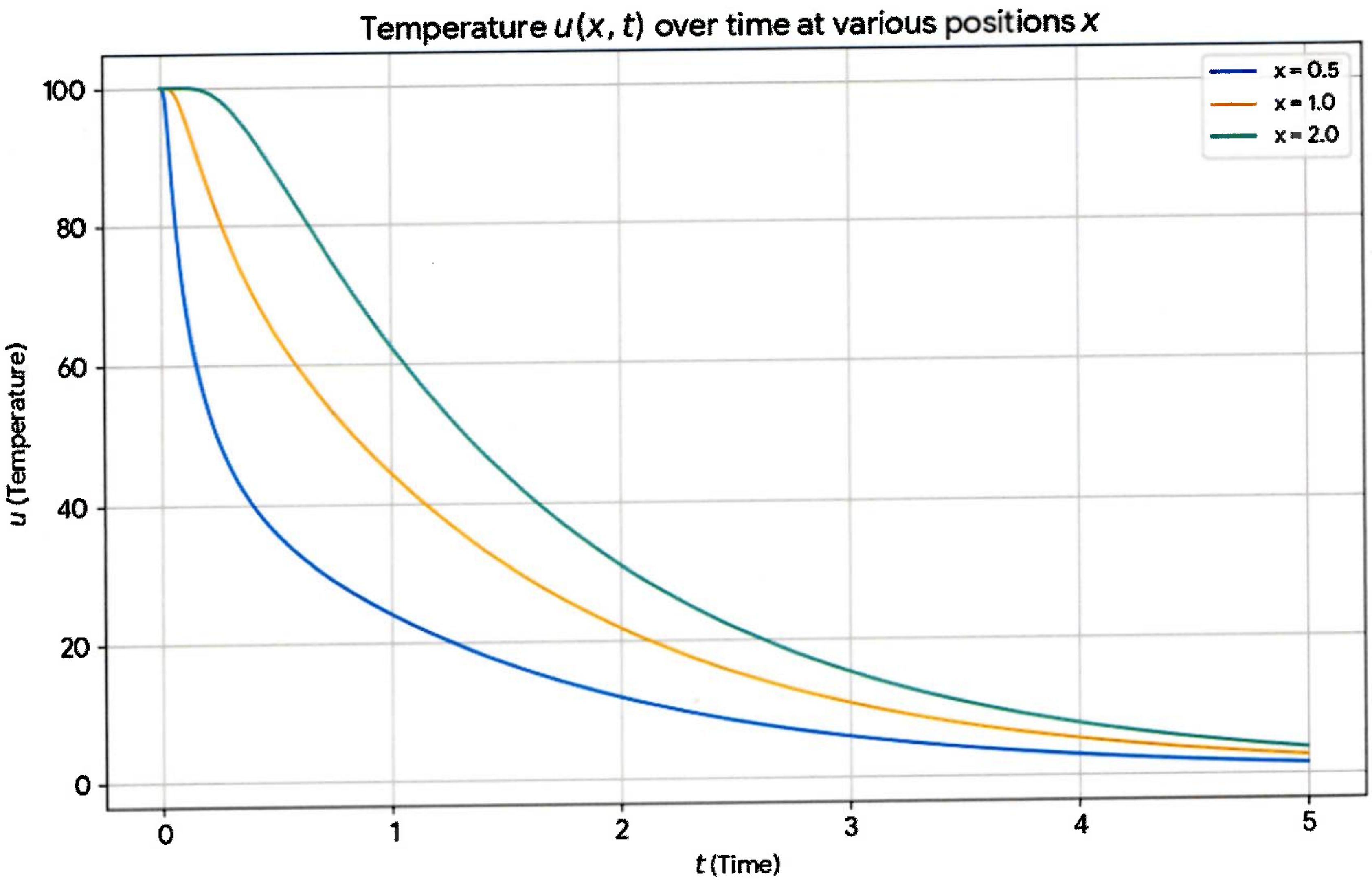
$$u(2,3) = \sum_{n=1}^{\infty} \frac{200\left[1-(-1)^n\right]}{n\pi} e^{-1.15 n^2 \pi^2 \cdot 3/16} \sin\left(\frac{n\pi \cdot 2}{4}\right)$$ <u>infinite</u> sum

in practice, negative exponential → fast convergence → few terms needed for good approx.

1-term approx ($n=1$ only) → 15.16 °C


Surface Plot of Heat Equation Solution $u(x, t)$
x (Position)
t (Time)
u (Temperature)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0
1
2
3
4
5
0
20
40
60
80
100
20
40
60
80

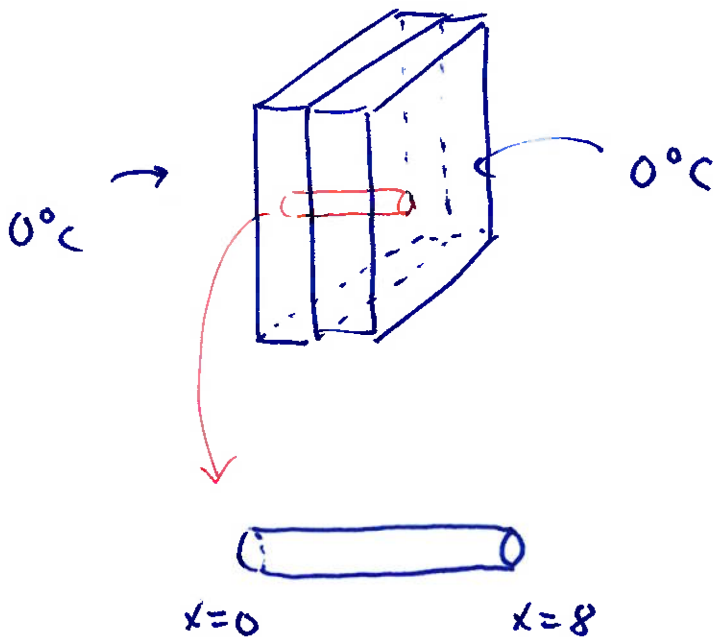

Temperature Profile u(x, t) at various times t
u (Temperature)
x (Position)
100
80
60
40
20
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
t = 0
t = 0.1
t = 0.5
t = 1.0
t = 2.0
t = 5.0



Temperature $u(x, t)$ over time at various positions x
x = 0.5
x = 1.0
x = 2.0
100
80
60
40
20
0
u (Temperature)
0
1
2
3
4
5
t (Time)

two slabs stuck together, both outer faces at 0°C

left slab heated to 50°C at $t=0$, right slab at 100°C



left 50°C

right 100°C

4 cm thick each

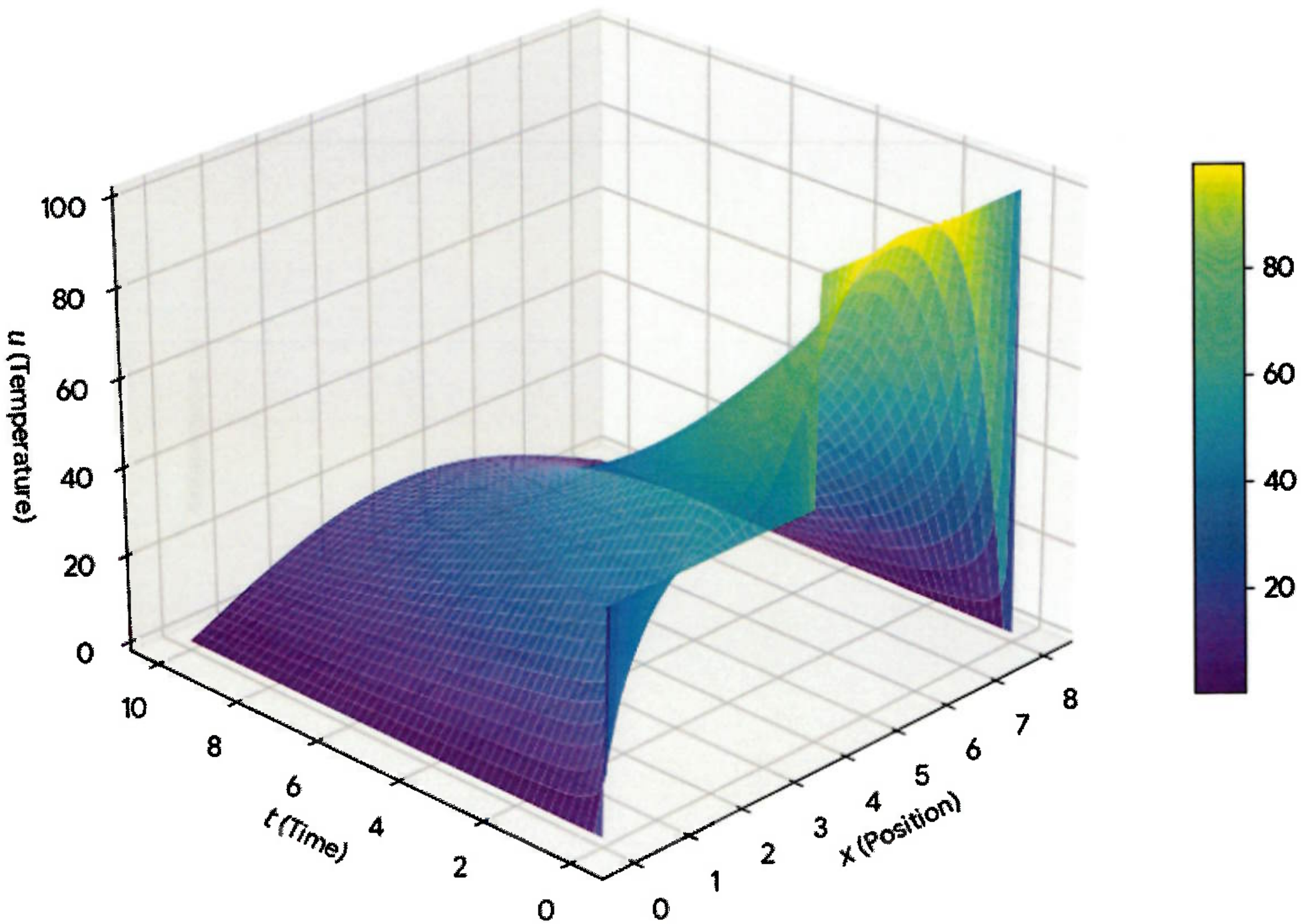
$L = 8$

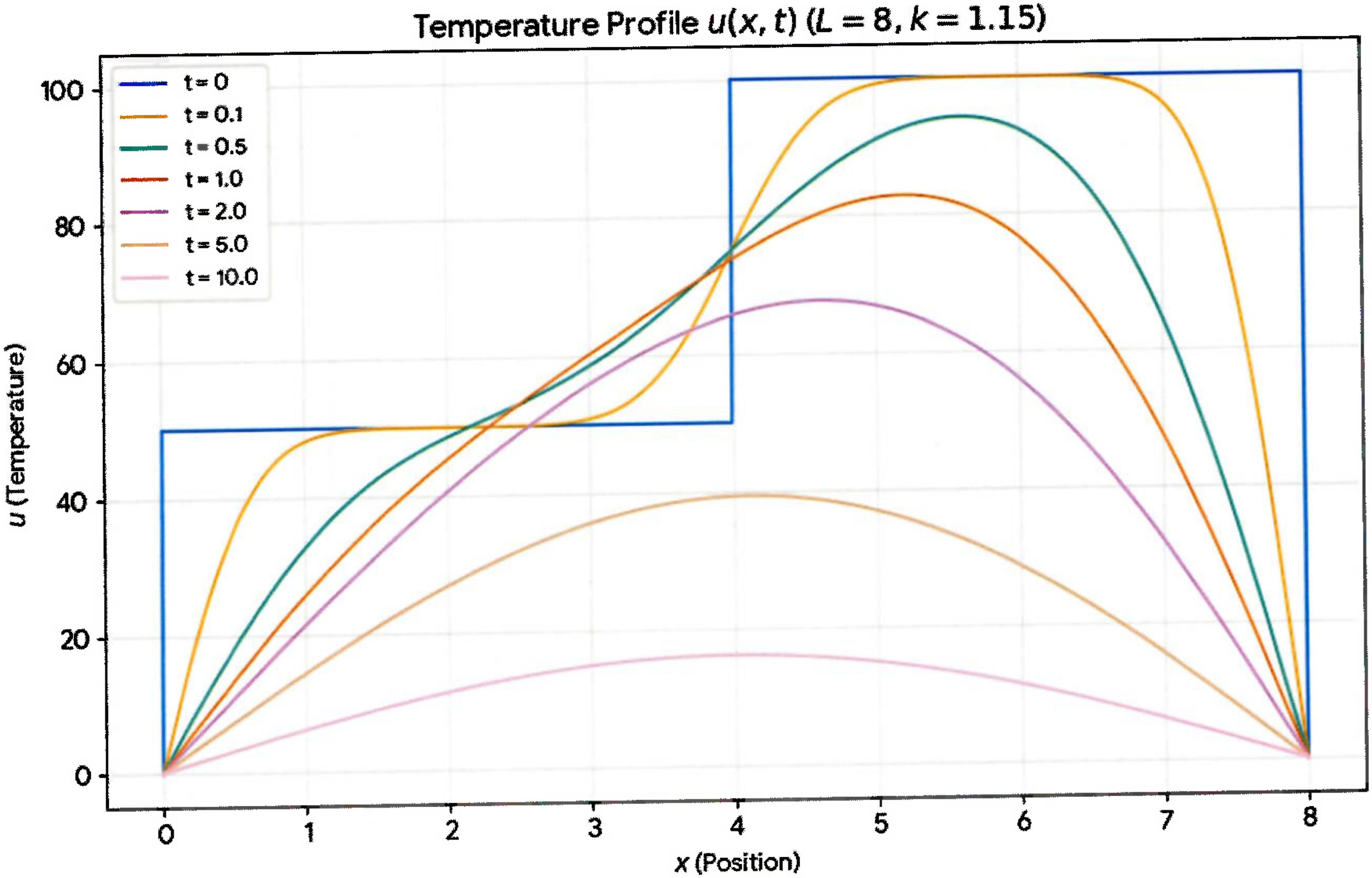
initial temp: $f(x) = \begin{cases} 50 & 0 < x < 4 \\ 100 & 4 < x < 8 \end{cases}$

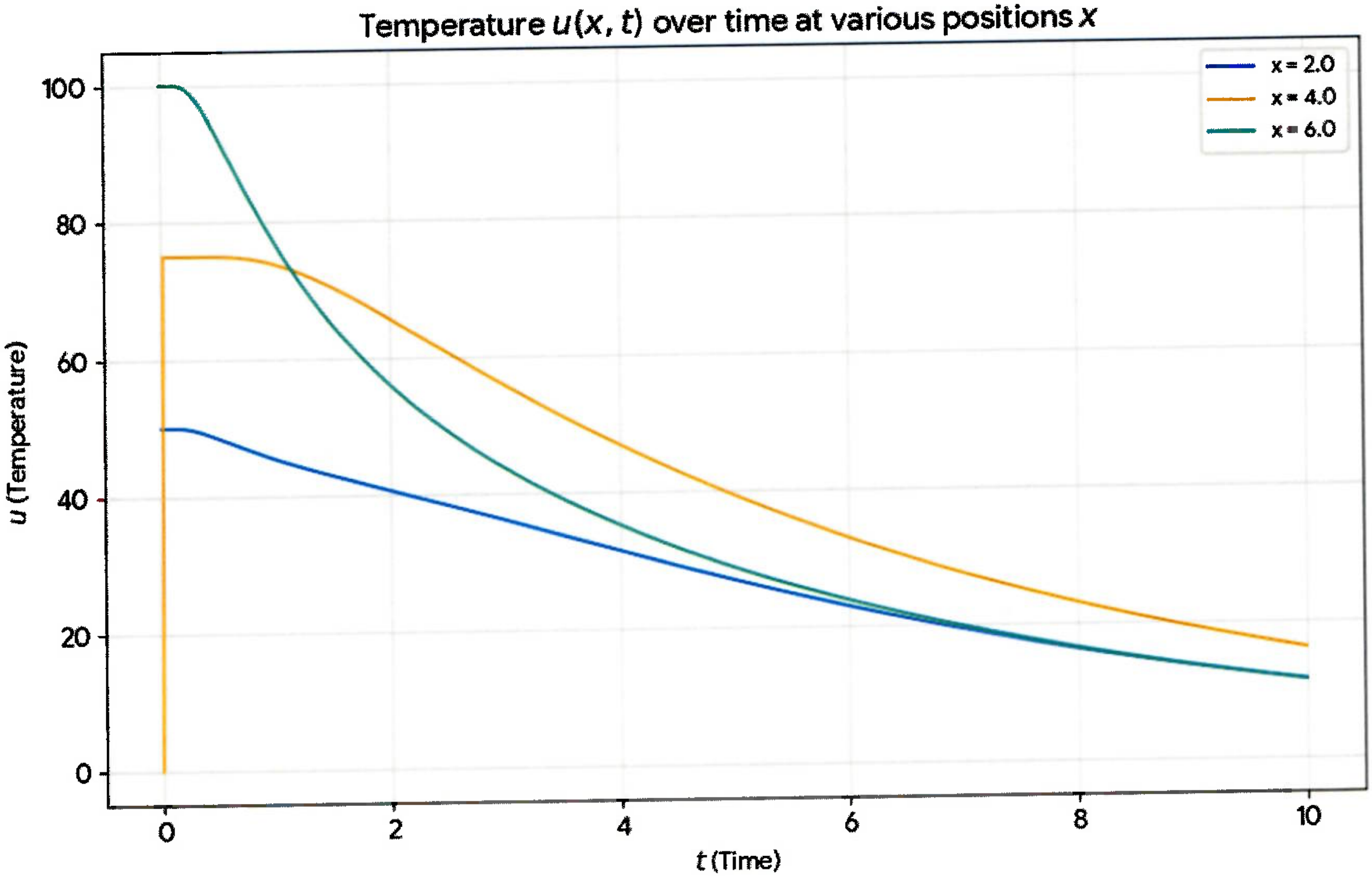
⋮

$$u(x,t) = \sum_{n=1}^{\infty} \frac{100}{n\pi}\left[1 + \cos\left(\frac{n\pi}{2}\right) - 2(-1)^n\right] e^{-1.15 n^2 \pi^2 t/64} \sin\left(\frac{n\pi x}{8}\right)$$

Surface Plot $u(x, t)$ ($L = 8$, $k = 1.15$, non-uniform initial)




Temperature Profile $u(x, t)$ ($L = 8$, $k = 1.15$)
t = 0
t = 0.1
t = 0.5
t = 1.0
t = 2.0
t = 5.0
t = 10.0
u (Temperature)
x (Position)
0
20
40
60
80
100
0
1
2
3
4
5
6
7
8



Temperature $u(x, t)$ over time at various positions x
x = 2.0
x = 4.0
x = 6.0
100
80
60
40
20
0
u (Temperature)
0
2
4
6
8
10
t (Time)

now let's remove the ends at 0 temp constraint

$u(0,t) = T_1$

$u(L,t) = T_2$

if T_1, T_2 are not zero, the BC's are <u>nonhomogeneous</u>

$u_t = k u_{xx}$

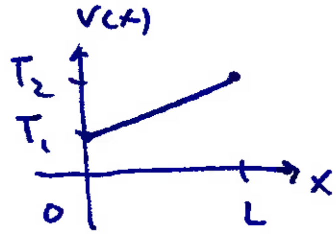
steady-state solution: $t \to \infty$ or $u_t = 0$ (time no longer changes solution)

$u_t = 0, \quad u_{xx} = 0 \to u = C_1 + C_2 x$

w/ $u(0) = T_1, \; u(L) = T_2$

$$u = \frac{T_2 - T_1}{L} x + T_1 = V(x)$$ steady-state solution

(a straight line)



what about the transient solution?

when time still matters

$$u_t = k u_{xx} \qquad v(x) = \frac{T_2 - T_1}{L} x + T_1$$

$u(0,t) = T_1$

$u(L,t) = T_2$

$u(x,0) = f(x)$

define $w(x,t) = u(x,t) - v(x)$ (original minus steady-state)

$w(0,t) = u(0,t) - v(0) = 0$

$w(L,t) = u(L,t) - v(L) = 0$

$w_t = u_t \qquad w_{xx} = u_{xx}$

$u_t = k u_{xx} \longleftrightarrow w_t = k w_{xx}$

} $w(x,t)$ has ~~non~~homogeneous BC's and $w_t = k w_{xx}$

use original solution for w

$$w(x,t) = \sum_{n=1}^{\infty} B_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = v(x) + \sum_{n=1}^{\infty} B_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u(x,t) = \left(\frac{T_2 - T_1}{L} x + T_2\right) + \sum_{n=1}^{\infty} B_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)}$$

initial condition: $u(x,0) = f(x)$

$$f(x) = \left(\frac{T_2 - T_1}{L} x + T_2\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\left[f(x) - v(x)\right] = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{sine series}$$

$$\boxed{B_n = \frac{2}{L} \int_0^L \left[f(x) - v(x)\right] \sin\left(\frac{n\pi x}{L}\right) dx}$$